# (Minimal Calc) Algebraic Proof that Monopolist Sets MR=MC

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To start, we assume the following:

The demand curve is of the form:

$$P(Q) = P(0) - \frac{P(0)}{N} * Q \tag{1}$$

for some $P(0) > 0$ being the maximum price any consumer is willing to pay and $N > 0$ being the total number of consumers. Note that the Marginal Revenue function has the same y-intercept as the demand function, P(0), but has half the x-intercept, N/2. Therefore, the Marginal Revenue function MR(Q) for the monopolist is:

$$MR(Q) = P(0) - \frac{2P(0)}{N} * Q \tag{2}$$

The monopolist has a total cost curve of the form:

$$C(Q) = \frac{c}{2} * Q^2 \tag{3}$$

for some cost parameter $c > 0$. Note that the only time we need to invoke calculus is here: taking a first derivative with respect to Q, we get that the marginal cost MC(Q) is:

$$MC(Q) = c * Q \tag{4}$$

To begin, we assume the monopolist maximizes profits by setting Marginal Revenue = Marginal Cost.

Therefore:

$$\begin{aligned} MR(Q) &= MC(Q) \\ P(0) - \frac{2P(0)}{N} * Q &= c * Q \\ P(0) &= [\frac{2P(0)}{N} + c] * Q \\ Q &= \frac{P(0)}{\frac{2P(0)}{N} + c} \\ Q &= \frac{NP(0)}{2P(0) + Nc} \end{aligned}$$

Thus the profit maximizing quantity Q* is:

$$Q^* = \frac{NP(0)}{2P(0) + Nc} \tag{5}$$

and the profit maximizing price P($Q^*$) is:

$$\begin{aligned} P(Q^*) &= P(0) - \frac{P(0)}{N} \frac{NP(0)}{2P(0) + Nc} \\ P(Q^*) &= P(0) - \frac{P(0)^2}{2P(0) + Nc} \end{aligned}$$

Finally, the maximum profit $\pi^*$ is:

$$\pi^* \equiv \pi(Q^*) = P(Q^*) * Q^* - \frac{c}{2}(Q^*)^2 \tag{6}$$

We're going to do a proof by contradiction. Suppose there exists some $\hat{\pi} > \pi^*$. Then there must exist some $\hat{Q} = Q^* + \epsilon$ for some $\epsilon \neq 0$ such that $\hat{\pi} \equiv \pi(\hat{Q}) > \pi^* = \pi(Q^*)$.

Then this means the following:

$$
\begin{aligned}
\hat{\pi} &= P(\hat{Q}) * \hat{Q} - \frac{c}{2}(\hat{Q})^2 \\
&= P(Q^* + \epsilon) * (Q^* + \epsilon) - \frac{c}{2}(Q^* + \epsilon)^2 \\
&= [P(0) - \frac{P(0)}{N}(Q^* + \epsilon)] * (Q^* + \epsilon) - \frac{c}{2}(Q^* + \epsilon)^2 \\
&= [(P(0) - \frac{P(0)}{N}Q^*) - (\frac{P(0)}{N}\epsilon)] * (Q^* + \epsilon) - \frac{c}{2}(Q^* + \epsilon)^2 \\
&= [P(Q^*) - (\frac{P(0)}{N}\epsilon)] * (Q^* + \epsilon) - \frac{c}{2}(Q^* + \epsilon)^2 \\
&= P(Q^*) * Q^* + \epsilon P(Q^*) - \epsilon\frac{P(0)}{N} * Q^* - \epsilon^2\frac{P(0)}{N} - \frac{c}{2}(Q^* + \epsilon)^2 \\
&= P(Q^*) * Q^* + \epsilon(P(0) - \frac{P(0)}{N}Q^*) - \epsilon\frac{P(0)}{N} * Q^* - \epsilon^2\frac{P(0)}{N} - \frac{c}{2}(Q^* + \epsilon)^2 \\
&= P(Q^*) * Q^* + \epsilon P(0) - 2\epsilon\frac{P(0)}{N} * Q^* - \epsilon^2\frac{P(0)}{N} - \frac{c}{2}(Q^* + \epsilon)^2 \\
&= P(Q^*) * Q^* + \epsilon P(0) - 2\epsilon\frac{P(0)}{N} * Q^* - \epsilon^2\frac{P(0)}{N} - \frac{c}{2}[(Q^*)^2 + 2Q^*\epsilon + \epsilon^2] \\
&= [P(Q^*) * Q^* - \frac{c}{2}(Q^*)^2] + \epsilon P(0) - 2\epsilon\frac{P(0)}{N} * Q^* - \epsilon^2\frac{P(0)}{N} - \frac{c}{2}[2Q^*\epsilon + \epsilon^2] \\
&= \pi^* + \epsilon P(0) - 2\epsilon\frac{P(0)}{N} * Q^* - \epsilon^2\frac{P(0)}{N} - \frac{c}{2}[2Q^*\epsilon + \epsilon^2] \\
&= \pi^* + \epsilon P(0) - 2\epsilon\frac{P(0)}{N} * (\frac{NP(0)}{2P(0) + Nc}) - \epsilon^2\frac{P(0)}{N} - \frac{c}{2}[2(\frac{NP(0)}{2P(0) + Nc})\epsilon + \epsilon^2] \\
&= [\pi^* - \epsilon^2(\frac{P(0)}{N} + \frac{c}{2})] + \epsilon P(0) - 2\epsilon\frac{P(0)}{N} * (\frac{NP(0)}{2P(0) + Nc}) - \frac{c}{2}[2(\frac{NP(0)}{2P(0) + Nc})\epsilon] \\
&= [\pi^* - \epsilon^2(\frac{P(0)}{N} + \frac{c}{2})] + \epsilon P(0) - \epsilon P(0) * [\frac{2P(0)}{2P(0) + Nc} + \frac{Nc}{2P(0) + Nc}] \\
&= [\pi^* - \epsilon^2(\frac{P(0)}{N} + \frac{c}{2})] + \epsilon P(0) - \epsilon P(0) \\
&= \pi^* - \epsilon^2(\frac{P(0)}{N} + \frac{c}{2})
\end{aligned}
$$

We thus have the following equation:

$$\hat{\pi} = \pi^* - \epsilon^2(\frac{P(0)}{N} + \frac{c}{2}) \tag{7}$$

But note that by the beginning assumptions, $P(0) > 0, N > 0, c > 0$, and since $\epsilon \neq 0$, $\epsilon^2(\frac{P(0)}{N} + \frac{c}{2}) > 0$, so $\hat{\pi} < \pi^*$, a contradiction.

[Q.E.D.]